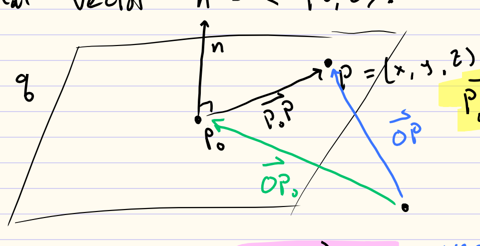
Meeting 5 Math 22

Section 9.5.3 Planes in Space

Recall, any plane is determined by a point $P_0 = (x_0, y_0, z_0)$ and a normal vector $n = \langle a, b, c \rangle$.



$\vec{P_0P} = \vec{OP} - \vec{OP_0} = \langle x-x_0, y-y_0, z-z_0 \rangle$

So P lies in q $\iff$ $n \cdot \vec{P_0P} = 0$ ← vector eq. of a plane

$\iff a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$ ← Scalar eq. of a plane

$\iff ax + by + cz = d$ where $d = n \cdot \langle x_0, y_0, z_0 \rangle$. ← alternative form of scalar eq.

Activity 9.5.4

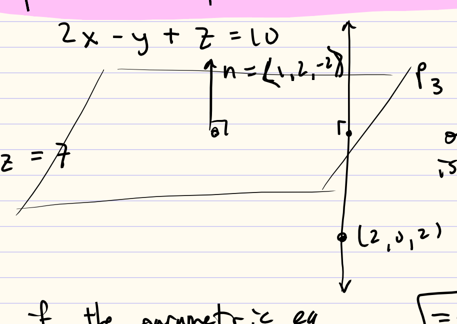
- Complete Activity 9.5.5 and discuss w/ your group.
- Class discussion.

a. $2x - y + z = 2$

b. No, $(2,0,2)$ doesn't satisfy the eq.

c. Two planes are parallel $\iff$ normal vectors are parallel

d.

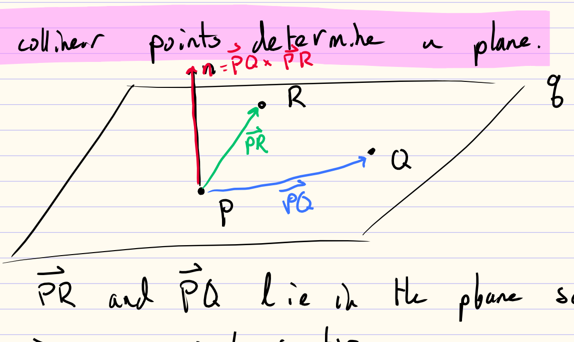


The direction of the line is n.

$r(t) = \langle 2,0,2 \rangle + tn = \langle 2+t, 0+2t, 2-2t \rangle$

e. Check if the parametric eq. satisfy the scalar eq for some value of t. ∎

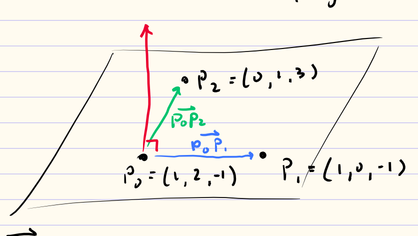
Any 3 non collinear points determine a plane.



The vectors $\vec{PR}$ and $\vec{PQ}$ lie in the plane so $n = \vec{PQ} \times \vec{PR}$ is a normal vector.

Activity 9.5.5

- Complete Activity 9.5.5 and discuss w/ your group.
- Class Discussion.



a. $\vec{P_0P_2} = \langle -1,-1,4 \rangle$ $\vec{P_0P_1} = \langle 0,-2,0 \rangle$

b. $n = \vec{P_0P_1} \times \vec{P_0P_2} = \langle -8, 0, -2 \rangle$

c. $-8x - 2z = -6$

d. Simplify eq: $-3x + 4y + 2z = -5$
$m = \langle -3, 4, 2 \rangle$ $P = (7/3, 0, 1)$ $Q = (0,0,-2.5)$

e. $\theta = \cos^{-1}\left(\frac{n \cdot m}{|n||m|}\right)$

End of Section 9.5

Section 9.6 Vector-Valued Functions

Reading Debrief

- Discuss Section 9.6 reading w/ your group.
- Are there any questions you want me to address?

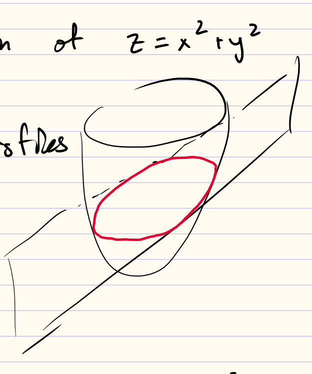
Questions?

- Find a vector valued function $\mathbf{r}(t)$ that describes the curve at the intersection of the paraboloid $z = 5x^2+5y^2$ and the (parabolic) cylinder $y = 5x^2$. Hint: use $x = t$ as the parameter. Plot your parameterization. You can compare with mine to see how you did: GeoGebra: Intersection of Paraboloid and Cylinder.

  Any point in the intersection satisfies both equation. Set $x(t) = t$. then $y(t) = 5t^2$.
  Then $z(t) = 5t^2 + 5(5t^2)^2 = 5t^2 + 125t^4$.
  So $r(t) = \langle t, 5t^2, 5t^2 + 125t^4 \rangle$.

- 9.6.4 (c). The level curve has the eq. $x^2 + y^2 = 25$. Since this is a circle, set $x(t) = 5\cos t$ $y(t) = 5\sin t$ and $z(t) = 25$. So
  $r(t) = \langle 5\cos t, 5\sin t, 25 \rangle$
- 9.6.4 (d) ($z = 25$ level curve for $f(x,y) = x^2 - y^2$)
  Set $x(t) = 5\sec t$ and $y(t) = 5\tan t$ and $z(t) = 25$.

Example Parameterize the intersection of $z = x^2 + y^2$ and $x + y + z = 1/2$.



Any pt. (x,y,z) on the curve satisfies both eq. So

$x^2 + x + y^2 + y = 1/2$

Complete the square:

$$\left(x^2 + x + \left(\tfrac{1}{2}\right)^2\right) + \left(y^2 + y + \left(\tfrac{1}{2}\right)^2\right) = 1/2 + 2\left(1/2\right)^2$$

$\Rightarrow (x+1/2)^2 + (y+1/2)^2 = 1$ ⇝ circle w/ radius 1 centered at $(-1/2, -1/2)$

Set $x(t) = \cos t - 1/2$, $y(t) = \sin t - 1/2$.
Then $z(t) = 1/2 - x(t) - y(t) = 3/2 - \cos t - \sin t$.

So $r(t) = \langle \cos t - 1/2, \sin t - 1/2, 3/2 - \cos t - \sin t \rangle$. ∎

Section 9.7 Derivatives and Integrals of Vector Valued Functions

Section 9.7.1 The Derivative

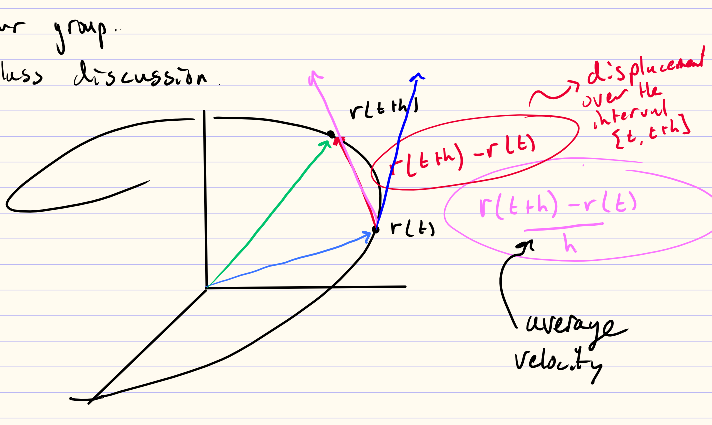
Definition The derivative of a vector-valued function $r(t)$ is defined to be

$$r'(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}$$

whenever the limit exists. The limit of a vector-valued function is defined componentwise. ∎

Activity 9.7.2

- Complete Activity 9.7.2 and discuss w/ your group.
- Class discussion.



d. The vector $\lim_{h \to 0} \frac{r(t+h) - r(t)}{h}$ is tangent to the curve.

Summary:

- $\frac{r(t+h) - r(t)}{h}$ is the average rate of change of an object traveling along the curve over the interval $[t, t+h]$. That is, average velocity over that interval.
- $r'(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}$ is therefore the instantaneous rate of change of the displacement of that object, i.e., instantaneous velocity.
- $r'(t)$ is the direction of the line tangent to the curve $r(t)$ at time t.
- $a(t) = v'(t) = r''(t)$ is the acceleration of the object at time t. ∎

Section 9.7.2 Computing Derivatives

Suppose $r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$. Then

$$r'(t) = \frac{d}{dt} r(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}.$$

Properties of Derivatives Let $f(t)$ be a differentiable real-valued function and let $r(t)$ and $s(t)$ be differentiable vector-valued functions

Then

1. $\frac{d}{dt}[r(t) + s(t)] = r'(t) + s'(t)$
2. $\frac{d}{dt}[f(t) r(t)] = f'(t) r(t) + f(t) r'(t)$
3. $\frac{d}{dt}[r(t) \cdot s(t)] = r'(t) \cdot s(t) + r(t) \cdot s'(t)$
4. $\frac{d}{dt}[r(t) \times s(t)] = r'(t) \times s(t) + r(t) \times s'(t)$
5. $\frac{d}{dt}[r(f(t))] = f'(t) r'(f(t))$. ∎

Activity 9.7.4

Compute $r'(t)$ for the following vector-valued functions.

a) $r(t) = \langle \cos t, t\sin t, \ln t \rangle$
$r'(t) = \langle \frac{d}{dt}\cos t, \frac{d}{dt} t\sin t, \frac{d}{dt}\ln t \rangle$
$= \langle -\sin t, \sin t + t\cos t, \frac{1}{t} \rangle$

c) $r(t) = \langle \tan t, \cos t^2, te^{-t} \rangle$
$r'(t) = \langle \frac{d}{dt}\tan t, \frac{d}{dt}\cos t^2, \frac{d}{dt} te^{-t} \rangle$
$= \langle \sec^2 t, (-\sin t^2) \cdot 2t, e^{-t} - te^{-t} \rangle$ ∎

Section 9.7.3 Tangent Lines

We expect that a smooth curve in space is locally linear, meaning we can approximate it locally w/ a line. To describe such a tangent line, we need a point on the line and its direction.

Activity 9.7.5

- Complete Activity 9.7.5 and discuss
- Class discussion

To summarize:

If $r(t)$ is a differentiable vector-valued function, then the tangent line to the curve at $t = a$ is given by

$$L(t) = r(a) + t r'(a).$$

Alternatively, you can use the parameterization

$$L(t) = r(a) + (t-a) r'(a).$$

In this one, $L(a) = 0$ so that the line "starts" at $t = a$. ∎